Hamburger's Theorem on $\zeta(s)$ and the Abundance Principle for Dirichlet Series with Functional Equations

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I Introduction

Ask any mathematician - indeed any number theorist - to state Hamburger's theorem; chances are the response will be something like, "Riemann's function $\zeta(s)$ is uniquely determined by its functional equation." In fact, this is correct, as far as it goes, but (as is often the case) closer examination show that it does not go nearly far enough.

Hecke grasped the subtleties inherent in Hamburger's theorem (1921) at least by 1944. In his final published paper [8], appearing that year, Hecke describes two versions of the theorem. I quote from the introduction to [8] (translation mine):

"The analytic function $\varphi(s)$ of the complex variable s is determined up to a constant by the following conditions: Put $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$".

1. With a suitable polynomial $P(s)$ suppose that $P(s)\varphi(s)$ is an entire function of finite genus.
2. Suppose $\varphi(s)$ satisfies the equation

$$R(s) = R\left(\frac{1}{2} - s\right).$$

3(a). Suppose that not only $\varphi(s)$, but also $\varphi(s/2)$, can be expanded in a Dirichlet series convergent somewhere:[1] $\varphi(s) = \sum_{n=1}^{\infty} b(n)n^{-2s}$. This condition can also be replaced by

3(b). Suppose that the only pole allowed for $\varphi(s)$ is $s = 1/2^2$; but we assume only the expressibility of $\varphi(s)$ itself as a Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$, not that of $\varphi(s/2)$.

"Mr. Hamburger first proved that $\varphi(s)$ is uniquely determined by 1, 2, 3 (a) and thus = const. $\zeta(2s)$" [5]; "that also 1, 2, 3 (b) suffice I have proved within the framework of a general investigation, by means of reduction to the theory of certain automorphic functions [6]."

While Hamburger discovered and gave the first proof of the well-known theorem bearing his name, for our purposes Siegel's elegant proof, published one year after Hamburger's, has greater relevance. The two formulations described by Hecke are in some ways quite distinct, but Siegel's (and, indeed, Hamburger's) proof of the Hamburger version and Hecke's proof of his own version are closely linked by their common use of the Mellin transform or, more accurately, its inverse. The idea in both cases is to show that the inverse Mellin transform of the function $R(s)$ described by Hecke is a constant times $\vartheta(z) - 1$, where ϑ is the classical Jacobi function, given by

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z}, \tag{1.1}$$

for z in $\mathcal{H} = \{z \in \mathbb{C} | \mathrm{Im}\, z > 0\}$. From this fact it follows immediately that $\varphi(s) =$ const. $\zeta(2s)$.

As we shall observe in §II.2 (see especially (2.6)), the Hamburger condition 3(a), above, implies immediately that the inverse Mellin transform of $R(s)$ has the form

$$\sum_{n=1}^{\infty} b(n) e^{\pi i n^2 z}, \tag{1.2}$$

and thus has the general shape of $\vartheta(z) - 1$, even before the application of conditions 1 and 2. Siegel then shows that the two latter conditions lead to a "modular relation" for the series (1.2), not that of a modular form, but the more general relation of a "modular integral." (See (2.12), below, and §IV.2 for the definition.) As it turns out, in the presence of (1.2) this more general relation suffices to imply that the function defined by (1.2) equals const.$\{\vartheta(z) - 1\}$, and thus to conclude the proof.

The Hecke condition 3(b), on the other hand, gives nothing like (1.2) (only that $F(z)$, the inverse Mellin transform of $R(s)$, has the form $\sum_{n=1}^{\infty} \beta(n) e^{\pi i n z}$), but the severe restriction on the singularities of $\varphi(s)$ in 3(b), together with condition 2, implies instead that with a_0 suitably chosen, $F(z) + a_0$ is a modular form of weight $\frac{1}{2}$, possessing precisely the same transformation properties as does $\vartheta(z)$. Hecke then invokes a familiar uniqueness result on modular forms to conclude that $F(z) + a_0 = a_0 \vartheta(z)$, and thus that $\varphi(s) = 2a_0 \zeta(2s)$.

With these contrasting versions of Hamburger's theorem in mind, it appears natural to relax both the expressibility of $\varphi(s/2)$ as a Dirichlet series in 3(a) and the restriction on the poles of $\varphi(s)$ in 3(b), to conjecture that $\varphi(s)$ is uniquely determined by 1, 2 and

3. Suppose (only) that $\varphi(s)$ can be expanded in a Dirichlet series convergent somewhere.

While appealing, this conjectured "strong Hamburger's theorem" fails spectacularly. Indeed, [12, Theorem 1] presents the

Abundance Principle for Dirichlet Series with Functional Equation. *There exist infinitely many linearly independent Dirichlet series satisfying the conditions* 1, 2 *and* 3.

There are generalizations of this Principle. For detailed statements see §V.1, below, [12, §§I & V], and [13, Theorem 1].

The proofs of the Principle and its generalizations fall into two steps. The first is an application of the Riemann-Hecke correspondence, as extended by Bochner [1], to translate the question of existence of the desired Dirichlet series into a question of existence of modular integrals with equivalent properties (§IV.4). The second step is the construction, by means of Eichler's generalized Poincaré series [4, 11], of infinitely many linearly independent modular integrals of the appropriate kind. (See §V.2 for further details.)

II Siegel's proof

1. Preliminary observations. We begin by outlining Siegel's celebrated proof of Hamburger's theorem [17], above all because of its relevance to our point of view. Indeed, his proof foreshadows our approach to the Abundance Principle, featuring an application of the Riemann-Hecke-Bochner correspondence, fourteen years before Hecke developed it as a systematic link between modular forms, on the one hand, and Dirichlet series with functional equations, on the other [6, 7], and twenty-nine years before Bochner's generalization [1, 3] to modular integrals. (We stop short of claiming that Siegel's work antedates Riemann's invention of the correspondence in the latter's derivation of the functional equation of $\zeta(s)$ from the transformation formula of $\vartheta(z)$ under $z \to -\frac{1}{z}$. See (4.1), below.)

Furthermore, aside from Hurwitz's construction of the Eisenstein series E_2 of weight 2 on the full modular group [9], Siegel's proof contains the first (to my knowledge) published example of a modular integral with log-polynomial period function. (See (2.12) and §IV.3, below, for the definition.) It is certainly the first occurrence of a modular integral within the context of the Riemann-Hecke-Bochner correspondence.[3] (Of course, Siegel proceeds to show that the modular integral is a multiple of $\vartheta(z) - 1$ and thus, in fact, a modular form; however, this small irony does not diminish the point.) A model of mathematical insight and elegance, this proof is relevant to research today, notwithstanding the passage of three-quarters of a century.

The statement that Siegel-like Hamburger before him-proves differs from Hecke's description of it in two respects. It posits the existence of two Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad g(s) = \sum_{n=1}^{\infty} b_n n^{-s} \tag{2.1}$$

and a polynomial $P(s)$, such that

(i) $P(s)f(s)$ is an entire function of finite genus;

(ii) $f(s)$ converges absolutely for $\sigma = \operatorname{Re} s > 2 - \theta$ (some $\theta > 0$);

(iii) $g(s)$ converges absolutely for $\sigma > 1 + \alpha$ (some $\alpha > 0$);

(iv) $$\pi^{-s/2}\Gamma(s/2)f(s) = \pi^{-(\frac{1-s}{2})}\Gamma\left(\frac{1-s}{2}\right)g(1-s). \tag{2.2}$$

The conclusion: $f(s) = g(s) = \text{const.}\zeta(s)$.

The two ways in which Hecke rephrased the hypotheses (2.2) can be read in condition (2.2, iv). In Hecke the two Dirichlet series $f(s)$, $g(s)$ have been replaced by the single $\varphi(s)$. However, this apparent loss of generality is not significant since (2.2, iv) implies immediately that $R_1(s) = \pi^{-s/2}\Gamma(s/2)\{f(s)+g(s)\}$ and $R_2(s) = \pi^{(-s/2)}\Gamma(s/2)\{f(s)-g(s)\}$ satisfy, respectively, $R_1(s) = R_1(1-s)$ and $R_2(s) = -R_2(1-s)$, functional equations with the same Dirichlet series on both sides.

Hecke's second change in the functional equation amounts to a replacement of s by $2s$ in (2.2, iv), which then becomes

$$\pi^{-s}\Gamma(s)\hat{f}(s) = \pi^{-(\frac{1}{2}-s)}\Gamma\left(\frac{1}{2}-s\right)\hat{g}\left(\frac{1}{2}-s\right), \tag{2.3}$$

where $\hat{f}(s) = f(2s)$ and $\hat{g}(s) = g(2s)$. With $\hat{f} = \hat{g} = \varphi$ and $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$, (2.3) reduces to the Hecke formulation $R(s) = R(\frac{1}{2} - s)$. That $\varphi(s) = \hat{f}(s) = f(2s)$ accounts for the condition 3(a) in Hecke's description of Hamburger's version, the condition which, in his own formulation, he replaces by 3(b), the restriction on the poles of $\varphi(s)$.

2. Outline of the Proof. Siegel's proof is considerably more direct than that of Hamburger, using only the familiar formulae

$$e^{-y} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} y^{-s}\Gamma(s)ds,\ y > 0 \tag{2.4}$$

(the inverse Mellin transform of $\Gamma(s)$ is e^{-y}) and

$$\int_0^\infty e^{-a^2x-b^2/x}\frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{a}e^{-2ab}, \tag{2.5}$$

for $a > 0$, $b \geq 0$ (evaluation of the Bessel integral). The first step is the simple observation that (2.2, iv) implies $S_1 = S_2$ for $y > 0$, where

$$\begin{aligned} S_1 &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma(s/2)\pi^{-s/2}y^{-s/2}ds, \\ S_2 &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(\frac{1-s}{2})}y^{-s/2}ds. \end{aligned}$$

From (2.4) and an easily-justified interchange of sum and integral it follows that

$$S_1 = 2\sum_{n=1}^{\infty} a_n e^{-\pi n^2 y}. \tag{2.6}$$

Thus S_1 is the exponential series

$$2\sum_{n=1}^{\infty} a_n e^{\pi i n^2 z},\ \mathrm{Im}\, z > 0,$$

evaluated on the positive imaginary axis $z = iy$, $y > 0$. It is this series that must be proved equal to $\alpha(\vartheta(z) - 1)$, with $\alpha \in \mathbb{C}$. (This equation is equivalent to: $a_n = \alpha$ for all $n \geq 1$. The same is true of the equation $f(s) = \alpha\zeta(s)$.)

The conditions (2.2, i, ii), the functional equation (2.2, iv), Stirling's formula and the Phrágmen-Lindelöf principle combine to give an estimate on the growth of $g(1-s)$ in the vertical strip $-\alpha - 1 \leq \sigma \leq 2$, an estimate sufficiently strong to make possible an application of the residue theorem to S_2 in the infinite strip $-\alpha - 1 \leq \sigma \leq 2$, $|t| \geq T_0 > 0$ (T_0 sufficiently large). This yields

$$S_2 = \frac{1}{2\pi i}\int_{-\alpha-1-i\infty}^{-\alpha-1+i\infty} g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(\frac{1-s}{2})}y^{-s/2}ds + \sum_{\nu=1}^{m} R_\nu(y), \tag{2.7}$$

where the $R_\nu(y)$ are the residues of the integrand at the poles in the region $-\alpha - 1 < \sigma < 2$.

The Dirichlet series representation (2.1) of $g(s)$ and (2.4) together show that the integral in (2.7) is equal to

$$\frac{2}{\sqrt{y}}\sum_{n=1}^{\infty} b_n e^{-\pi n^2/y},\ y > 0.$$

On the other hand, calculation of the residues of the integrand leads to

$$\sum_{\nu=1}^{m} R_\nu(y) = \sum_{\nu=1}^{m} y^{-s_\nu/2} Q_\nu(\log y) = Q(y), \tag{2.8}$$

where the Q_ν are polynomials and the s_ν are the poles of the integrand in $-\alpha - 1 < \sigma < 2$. (The conditions (2.2, ii) and (2.2, iv) imply that Re $s_\nu \le 2 - \theta$, for $1 \le \nu \le m$.) Then, $S_1 = S_2$, (2.6) and (2.7) together yield the fundamental transformation property

$$2\sum_{n=1}^{\infty} a_n e^{-\pi n^2 y} = \frac{2}{\sqrt{y}}\sum_{n=1}^{\infty} b_n e^{-\pi n^2/y} + Q(y). \tag{2.9}$$

At this juncture Siegel introduces another integral transform, multiplying both sides of (2.9) by $e^{-\pi t y^2}$, with fixed $t > 0$, and integrating on y from 0 to ∞. Absolute convergence justifies termwise integration on both sides of (2.9); the application of (2.5) on the right-hand side leads to

$$\sum_{n=1}^{\infty} a_n \left(\frac{1}{t+ni} + \frac{1}{t-ni}\right) - \pi t H(t) = 2\pi \sum_{n=1}^{\infty} b_n e^{-2\pi n t}, \tag{2.10}$$

where $H(t) = \sum_{\nu=1}^{m} t^{s_\nu - 2} H_\nu(\log t)$, with H_ν a polynomial. Since the identity (2.10) clearly extends to the half-plane Re $t > 0$, Siegel can exploit the periodicity of the right-hand side of (2.10) and the singularities of the left-hand side, to conclude that $a_n = a_1$ for all $n \ge 1$, thus that $f(s) = a_1\zeta(s)$. This completes the proof.

Remark: If we put

$$F(z) = 2\sum_{n=1}^{\infty} a_n e^{\pi i n^2 z},\ G(z) = 2\sum_{n=1}^{\infty} b_n e^{\pi i n^2 z},$$

then (2.9) becomes

$$F(z) = (z/i)^{-1/2} G(-1/z) + Q(z/i), \tag{2.11}$$

on the positive imaginary axis: $z = iy$, $y > 0$. By the principle of analytic continuation, (2.11) holds in all of $\mathcal{H}$. Thus, in $\mathcal{H}$, $K(z) = F(z) + G(z)$ satisfies

$$K(z) = (z/i)^{-1/2} K(-1/z) + Q(z/i) - (z/i)^{-1/2} Q(i/z). \tag{2.12}$$

With $Q(z/i)$ of the form (2.8), the same is true of $(z/i)^{-1/2}Q(i/z)$, so that $Q(z/i) - (z/i)^{-1/2}Q(i/z)$ is a "log-polynomial sum." Consequently, $K(z)$ is a "modular integral with log-polynomial period function." (See §IV, 3 below.)

III Hecke's Proof

The beginning of the Hecke proof in essence follows that of Siegel, modified only to take account of the modified hypotheses. In place of the functional equation (2.2, iv) linking the Dirichlet series f and g, Hecke assumes

$$\pi^{-s}\Gamma(s)\varphi(s) = \pi^{-(\frac{1}{2}-s)}\Gamma\left(\frac{1}{2}-s\right)\varphi\left(\frac{1}{2}-s\right), \tag{3.1}$$

with φ a Dirichlet series: $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. In place of Siegel's assumption (2.2, i), which permits f to have an arbitrary finite number of poles in the s-plane, Hecke imposes the condition that $(s-\frac{1}{2})\varphi(s)$ can be continued to an entire function of finite genus. That is to say, in Hecke's formulation the polynomial $P(s) = s - 1/2$; this is what Hecke actually intends in his condition 3(b). (See §I, above.)

Siegel's technique, employing the inverse Mellin transform, is equally effective under Hecke's modified assumptions. Using this procedure, Hecke obtains, in place of Siegel's transformation law (2.9),

$$\sum_{n=0}^{\infty} c_n e^{-\pi n y} = \frac{1}{\sqrt{y}} \sum_{n=0}^{\infty} c_n e^{-\pi n/y}, \tag{3.2}$$

where $c_n, n \geq 1$, are the coefficients in the expression of $\varphi(s)$ as a Dirichlet series and $c_0 = \operatorname{Res}_{s=1/2} \varphi(s)$. (Comparison shows obvious changes from (2.9), upon which we shall comment shortly.)

Putting $L(z) = \sum_{n=0}^{\infty} c_n e^{\pi i n z}$ permits us to rewrite (3.2) as

$$L(z) = \frac{1}{\sqrt{-iz}} L(-1/z),\ z = iy,\ \ y > 0. \tag{3.3}$$

Since $\varphi(s)$ is assumed to converge in some right half-plane, the coefficients c_n have at worst polynomial growth in n. This implies that $L(z)$ is holomorphic in $\mathcal{H}$, so (3.3) holds in all of $\mathcal{H}$. The definition of $L(z)$ shows further that $L(z+2) = L(z)$, and this, together with (3.3) and the growth condition on the coefficients c_n, implies that $L(z)$ is an entire modular form of weight 1/2 on Γ_ϑ, the subgroup of index 3 in $SL(2, Z)$ generated by $z \to z+2$ and $z \to -1/z$. (See §IV.2, below. Recall that the full modular group $SL(2, Z)$ is generated by $z \to z+1$ and $z \to -1/z$.)

Since $L(z)$ has precisely the same transformation properties with respect to the generators of Γ_ϑ as does the Jacobi ϑ-function (1.1) (see (4.1), below), Hecke can complete his proof simply by comparing $L(z)$ with $\vartheta(z)$. It turns out that $L(z)/\vartheta(z)$ is entire modular *function* (i.e. modular form of weight 0) on Γ_ϑ, and thus a constant. But $L(z) = \text{const.}\ \vartheta(z)$ leads directly to $\varphi(s) = \text{const.}\zeta(s)$. This concludes the proof.

Remarks: The transformation law (3.3) differs from (2.12) in two essential respects:

(i) The series defining $K(z) = F(z) + G(z)$ is supported on integral squares, while the exponents in the series defining $L(z)$ are linear in n.

(ii) The "period function" $Q(z/i) - (z/i)^{-1/2}Q(i/z)$ appearing on the right-hand side of (2.12) is not present in (3.3)

That $K(z)$ is not *a priori* a modular form on Γ_ϑ, but only a modular integral with period function, makes Hecke's proof unavailable under the conditions Siegel imposes (notwithstanding that, as a consequence of his proof of $f(s) = g(s) = a_1\zeta(s)$, $K(z) = 2a_1(\vartheta(z) - 1)$, so $K(z) + 2a_1$ is indeed a modular form).

Equally, Siegel's proof fails if applied to Hecke's case. For, multiplying both sides of (3.2) by $e^{-\pi t^2 y}$ and integrating on y from 0 to ∞ yields

$$\sum_{n=0}^{\infty} \frac{c_n t}{\pi(t^2+n)} = \sum_{n=0}^{\infty} c_n e^{-2\pi t\sqrt{n}} \tag{3.4}$$

in place of (2.10). Siegel's derivation of Hamburger's theorem from (2.10) breaks down if we start instead with (3.4).

IV Modular Integrals and the Riemann-Hecke-Bochner Correspondence

The term "modular integral" has arisen several times above, most prominently in the discussion of Siegel's proof of Hamburger's theorem (§II). In this section we define the notion, important here for its role in explaining the origin of the Abundance Principle for Dirichlet Series (§I).

The ideas we shall introduce apply to the entire class of finitely generated discrete groups Γ acting on $\mathcal{H}$, of finite or infinite hyperbolic area. However, for the most part we restrict the discussion to $\Gamma = \Gamma_\vartheta$, the subgroup of $SL(2, Z)$ generated by $z \to z+2$, $z \to -1/z$. (See §III, above.) The group Γ_ϑ is so called because of its connection with the Jacobi function defined by (1.1): $\vartheta(z)$ is an entire modular form of weight $\frac{1}{2}$ on Γ_ϑ. That is, for z in $\mathcal{H}$,

$$\vartheta(z+2) = \vartheta(z),\ \vartheta(-1/z) = e^{-\pi i/4} z^{1/2} \vartheta(z), \tag{4.1}$$

and $\vartheta(z)$ has bounded behaviour at the two parabolic points of a fundamental region $\mathcal{R}$ for Γ_ϑ. ($\mathcal{R}$ can be so chosen that the two parabolic points are $i\infty$ and -1. The expansion (1.1) expresses the behaviour at $i\infty$; there is a similar expansion at -1. See [10, Theorem 13, 46].)

1. Multiplier systems and period cocycles for Γ_ϑ. Let k be a real number and v a "multiplier system" for the weight k and the group Γ_ϑ. This means that v is a function on the group Γ_ϑ - thought of as a *matrix group* - such that

$$\begin{aligned} &\text{(i)}\ \ |v(M)| = 1 \text{ for all } M \text{ in } \Gamma_\vartheta; \\ &\text{(ii)}\ \ v(M_3)(c_3 z + d_3)^k = v(M_1)v(M_2)(c_1 M_2 z + d_1)^k (c_2 z + d_2)^k. \end{aligned} \tag{4.2}$$

The identity (4.2, ii) is required to hold for all z in $\mathcal{H}$ and M_1, M_2 in Γ_ϑ, with $M_3 = M_1 M_2$ and $M_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}$, $1 \le i \le 3$. It is not too hard to show from (4.2, ii) that v is a character on the matrix group Γ_ϑ when $k \in Z$, and a character on the linear fractional transformation group Γ_ϑ when $k \in 2Z$.

With k real, v a fixed multiplier system in weight k for Γ_ϑ and f a function defined on $\mathcal{H}$, we introduce the *stroke* (or *slash*) *operator*

$$(f|_k^v M)(z) = \bar{v}(M)(cz+d)^{-k} f(Mz), M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\vartheta. \tag{4.3}$$

With this notation, condition (4.2, ii) is equivalent to

$$f|_k^v M_1 M_2 = (f|_k^v M_1)|_k^v M_2, \tag{4.4}$$

for M_1, M_2 in Γ_ϑ and any f defined on $\mathcal{H}$.

Suppose f is a function holomorphic in $\mathcal{H}$; define the functions $q_M(z) = q(M; z)$, $M \in \Gamma_\vartheta$, as follows:

$$\bar{v}(M)(cz+d)^{-k} f(Mz) = f(z) + q_M(z), \tag{4.5, i}$$

with $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\vartheta$. This can be rewritten as

$$f|M = f + q_M, M \in \Gamma_\vartheta. \tag{4.5, ii}$$

(Note the abbreviation $f|M$ for $f|_k^v M$.) The q_M are called the *period functions* of f relative to $\{\Gamma_\vartheta, k, v\}$. From (4.4) the *cocycle condition* for $\{q_M | M \in \Gamma_\vartheta\}$ follows directly:

$$q_{MN} = q_M|N + q_N, \text{ for } M, N, \in \Gamma_\vartheta. \tag{4.6}$$

A collection of functions $\{q_M | M \in \Gamma_\vartheta\}$ satisfying (4.6) is called a *cocyle for (*or *relative to)* $\{\Gamma_\vartheta, k, v\}$.

The condition (4.5) above does not restrict f in any way, since q_M is defined to be $f|M - f$. To construct a meaningful theory we impose restrictions upon the q_M, suited to the purpose at hand. In the present context it is essential to assume that the q_M lie in $\mathcal{P}$, the collection of all functions f holomorphic in $\mathcal{H}$, subject to the growth condition

$$|f(z)| \le K(|z|^\alpha + y^{-\beta}), y = \text{Im } z > 0, \tag{4.7}$$

for some constants $K, \alpha, \beta > 0$. Note that $\mathcal{P}$ is an algebra over the complex field $\mathbb{C}$; moreover, it is preserved under differentiation, integration and the stroke operator (4.3), with $M \in SL(2, \mathbb{R})$.

2. Modular integrals. Assume that $\{q_M\}$ is a cocycle relative to $\{\Gamma_\vartheta, k, v\}$ such that $q_M \in \mathcal{P}$, for all $M \in \Gamma_\vartheta$. Suppose that f is holomorphic in $\mathcal{H}$ and satisfies (4.5). Standard arguments using (4.5) imply the existence of "Fourier expansions" for f at the parabolic cusps $i\infty$ and -1, of Γ_ϑ. (See, for example, [10, Chapter 2, 17–23].) Let $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The expansion at $i\infty$ has the form

$$f(z) = \rho_0(z) + \sum_{n=-\infty}^{\infty} a_n e^{\pi i(n+\kappa)z}, \text{ Im } z > 0, \tag{4.8a}$$

with complex coefficients a_n. Here ρ_0 is a function in $\mathcal{P}$ determined by $q(S^2; z)$ and κ derives from the multiplier system $v : v(S^2) = e^{2\pi i\kappa}, 0 \leq \kappa < 1$. The expansion at -1 has the form

$$f(z) = \rho_1(z) + (z+1)^{-k} \sum_{n=-\infty}^{\infty} b_n \exp\{-2\pi i(n+\mu)/(z+1)\}, \tag{4.8b}$$

with complex b_n and $\rho_1 \in \mathcal{P}$ determined by $q(S^{-2}T; z)$; μ is defined by $v(S^{-2}T) = e^{2\pi i\mu}$, $0 \leq \mu < 1$.

Assume that the expansions (4.8) are both left-finite. Then we call f a *modular integral with respect to Γ_ϑ, of weight k and multiplier system v, with period functions* (or *period cocycle*) $\{q_M | M \in \Gamma_\vartheta\}$. If no a_n, b_n with $n < 0$ occur in the expansions (4.8) we call f an *entire* modular integral. If $q_M = 0$ for all M in Γ_ϑ, f is a *modular form* (*entire* modular form) rather than a modular integral (*entire* modular integral).

In §IV.4 we shall introduce the Riemann-Hecke-Bochner (R-H-B) correspondence, which plays an essential role in the derivation of the Abundance Principle (§I, above). To gain a measure of flexibility we state the correspondence not merely for Γ_ϑ, but for the entire class of "Hecke triangle groups". With $\lambda > 0$ let $S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ (hence $S_1 = S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S_2 = S^2$). Then the *Hecke group* G_λ is defined by

$$G_\lambda = \langle S_\lambda, T \rangle. \tag{4.9}$$

(Note that $G_1 = \Gamma(1) = SL(2, Z)$ and $G_2 = \Gamma_\vartheta$.)

Hecke [7] has shown that G_λ is discrete if and only if (i) $\lambda \geq 2$ or (ii) $\lambda = 2\cos\pi/p$, with $p \in Z$, $p \geq 3$. When $\lambda \geq 2$, G_λ has the single relation $T^2 = I$; when $\lambda = 2\cos\pi/p$, G_λ has the two relations $T^2 = (S_\lambda T)^p = I$. When $\lambda \leq 2$, G_λ has a fundamental region of finite hyperbolic area. For $\lambda > 2$, by contrast, this area is infinite.

One can define "period cocyle for G_λ" and "(entire) modular integral" by analogy with the definitions given above for $\Gamma_\vartheta (\lambda = 2)$. For general $\lambda > 0$, the expansion (4.8a) is replaced by

$$f(z) = \rho_0(z) + \sum_{-\infty}^{\infty} a_n e^{2\pi i(n+\kappa)/\lambda}, \ \mathrm{Im}\, z > 0. \tag{4.10}$$

If $\lambda \neq 2$, the point -1 is not a parabolic cusp, so there is no analogue of (4.8b). The definitions of "modular integral" and "entire modular integral" of course entail the same restrictions on the expansion (4.10) as in the case $\lambda = 2$.

3. Modular integrals with log-polynomial period. Our application to the Abundance Principle for Dirichlet series requires period cocycles $\{q_M | M \in G_\lambda\}$ satisfying conditions far more stringent than $q_M \in \mathcal{P}$. To describe these conditions we introduce *log-polynomial sums*, functions of the form

$$q(z) = \sum_{1 \leq j \leq J} z^{\alpha_j} \sum_{0 \leq t \leq M(j)} \beta(j, t)(\log z)^t, \tag{4.11}$$

where the exponents α_j and the coefficients $\beta(j, t)$ are arbitrary complex numbers. The t are nonnegative integers. Note that a log-polynomial sum is holomorphic in $\mathbb{C}\backslash\{iy|y \leq 0\}$; in particular, $q(z)$ defined by (4.11) is holomorphic in $\mathcal{H}$.

We say that the modular integral f on Γ_ϑ (respectively, G_λ) has *log-polynomial* period, provided $q(S^2; z) \equiv 0$ (respectively, $q(S_\lambda; z) \equiv 0$) and $q(T; z)$ is a log-polynomial sum, for the generators S^2 (respectively, S_λ) and T, in the modular transformation law for f ((4.5) or its analogue for G_λ). Note that $q(T; z^2)$ and $q(S^2; z)(q(S_\lambda; z))$ are in $\mathcal{P}$. Since $\Gamma_\vartheta = \langle S^2, T\rangle$ (respectively, $G_\lambda = \langle S_\lambda, T\rangle$), it follows that $q_M \in \mathcal{P}$ for all $M \in \Gamma_\vartheta$ (respectively, $M \in G_\lambda$), by the closure properties of $\mathcal{P}$ and the cocycle condition (4.6) (or its analogue for G_λ). The cocycle condition implies as well that $q_T|T + q_T = 0$, since $T^2 = I$.

The significance for us of modular integrals with log-polynomial period is this: *By the Riemann-Hecke-Bochner correspondence* (§IV.4, below), *the Abundance Principle for Dirichlet Series with Functional Equation is equivalent to the existence of infinitely many linearly independent entire modular integrals on Γ_ϑ, with log-polynomial period.*

4. The Riemann-Hecke-Bochner correspondence. Before stating the correspondence we make a few observations about the expansion (4.8a). If f is an entire modular integral on Γ_ϑ, then by our definition the expansion has the form

$$f(z) = \rho_0(z) + \sum_{n=0}^{\infty} a_n e^{\pi i(n+\kappa)z}.$$

If, in addition, $q(S^2, z) \equiv 0$ (as is the case when f has log-polynomial period), the expansion has the form $f(z) = \sum_{n=0}^{\infty} a_n e^{\pi i(n+\kappa)z}$.

We note further that the only relation in Γ_ϑ, $T^2 = I$, does not involve the generator S^2. Thus, in any weight k we can determine a multiplier system v on Γ_ϑ by choosing $v(S^2) = e^{2\pi i\kappa}$, with $0 \leq \kappa < 1$, but κ otherwise arbitrary, and putting $v(T) = C$, with C chosen to respect the relation $T^2 = I$. By (4.2, ii), this means that $v(T)$ has one of the four values $\pm e^{-\pi ik/2}$, $\pm ie^{-\pi ik/2}$; however, only the two values $\pm e^{-\pi ik/2}$ give rise to nontrivial modular forms or modular integrals on Γ_ϑ. Now, given M in Γ_ϑ we write M as a word in S^2 and T, and determine $v(M)$ from $v(S^2)$, $v(T)$ by applying the consistency condition (4.2, ii).

In the statement of the correspondence we choose $\kappa = 0$, so $v(S^2) = 1$. Then the expansion at $i\infty$ of an entire modular integral f on Γ_ϑ assumes the form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{\pi inz}, \ \operatorname{Im} z > 0. \tag{4.12}$$

R-H-B Correspondence. Let k be real and C complex. Suppose $F(z)$ is holomorphic in the upper half-plane $\mathcal{H}$, defined there by an exponential series of the form (4.12), where the complex coefficients a_n satisfy the polynomial growth condition

$$a_n = \mathcal{O}(n^\gamma), \gamma > 0, n \longrightarrow \infty. \tag{4.13}$$

Let $\Phi(s)$ be the Mellin transform of $F(iy) - a_0$:

$$\Phi(s) = \int_0^\infty \{F(iy) - a_0\} y^s \frac{dy}{y} = \pi^{-s}\Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s}.$$

Then, we have the following.

1. The assertions (A) and (B) are equivalent:
   (A) $F(z)$ is an entire modular integral on Γ_ϑ, with log-polynomial period, of weight k and multiplier system v such that $v(S^2) = 1$, $v(T) = C$.
   (B) Φ can be continued analytically into the entire s-plane, except for a finite number of poles. Furthermore, Φ is bounded in each lacunary vertical strip,

   $$\sigma_1 \leq \operatorname{Re} s \leq \sigma_2, \, |\operatorname{Im} s| \geq t_0 > 0, \quad t_0 \text{ sufficiently large},$$

   and satisfies the functional equation

   $$\Phi(k - s) = e^{\pi ik/2} C \Phi(s). \tag{4.14}$$

2. Consider the modular relation included in condition (A):

   $$z^{-k} F(-1/z) = CF(z) + q_T(z), \tag{4.15}$$

   with $q_T(z)$ of the form (4.11). The term $\beta z^\alpha (\log z)^t$ ($\beta \neq 0, t \in Z, t \geq 0$) occurs in $q_T(z)$ if and only if $\Phi(s)$ has poles of order $t + 1$ at $s = \alpha + k$ and $s = -\alpha$. The only possible further singularities of Φ are simple poles at $s = 0$ and $s = k$.

Remarks: (i) By (4.14), $C = \pm e^{-\pi ik/2}$. (See the discussion immediately preceding the statement of the R-H-B correspondence.)

(ii) The correspondence, as stated here, differs somewhat from Bochner's original formulation in [1], which deals with generalized Dirichlet series rather than the ordinary Dirichlet series we have here; Bochner allows as well two distinct exponential series in the modular relation. Furthermore, Bochner's period function $q_T(z)$ in (4.15) is a "residual function" (in his terminology) rather than a sum of the form (4.11). (See also [3].) However, the sums (4.11) are residual in Bochner's sense, and a residual function appearing as a period function in a modular relation (4.15) necessarily has the form (4.11).

V The Abundance Principle for Dirichlet Series with Functional Equation

1. Detailed statement of results. In the Introduction we have stated the Abundance Principle in general terms, without explanatory details. We provide those now. Recall the conditions 1, 2, 3 of §I: Let $\varphi(s)$ be an analytic function and put $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$. Assume the following concerning $\varphi(s)$ and $R(s)$.

1. There exists a polynomial $P(s)$ such that $P(s)\varphi(s)$ is an entire function of finite genus.

2. $R(s) = R(\frac{1}{2} - s)$.
3. $\varphi(s)$ can be expanded in a Dirichlet series convergent in some right half-plane.

Theorem 1 *Let σ_0 be a real number $\geq \frac{1}{4}$. Let $\mathcal{A}(\sigma_0)$ be the space of rational functions $A(s)$ with poles restricted to the strip $\frac{1}{2} - \sigma_0 \leq \mathrm{Re}\, s \leq \sigma_0$ and satisfying the symmetry condition $A(\frac{1}{2} - s) = A(s)$. Let $\mathcal{A}_H(\sigma_0)$ be the subspace of A in $\mathcal{A}(\sigma_0)$ such that $R(s) - A(s)$ is entire for some $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$ satisfying 1, 2 and 3. Then for $A_1, \ldots, A_n \in \mathcal{A}(\sigma_0)$, with $n > [\frac{\sigma_0}{2} + \frac{3}{8}] + 2$, some nontrivial linear combination of $A_1, \ldots, A_n$ is in $\mathcal{A}_H(\sigma_0)$.*

Theorem 1 can be generalized to

Theorem 2 (Weight k Abundance Principle). *Let k be an arbitrary real number. For $\sigma_0 \geq k/2$, let $\mathcal{A}(\sigma_0; k)$ denote the space of rational functions $A(s)$ with poles restricted to the strip $k - \sigma_0 \leq \mathrm{Re}\, s \leq \sigma_0$ and such that $A(k - s) = A(s)$. Replace condition 2 by the functional equation*

$$R(s) = R(k - s). \tag{2_k}$$

Let $\mathcal{A}_H(\sigma_0; k)$ be the subspace of A in $\mathcal{A}(\sigma_0; k)$ such that $R(s) - A(s)$ is entire for some $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$ satisfying $1, 2_k$ and 3. Then for $A_1, \ldots, A_n \in \mathcal{A}(\sigma_0; k)$, with $n > N(\sigma_0, k)$, some nontrivial linear combination of $A_1, \ldots, A_n$ lies in $\mathcal{A}_H(\sigma_0; k)$. Here, $N(\sigma_0, k)$ is an explicit constant dependent only upon σ_0 and k.

For $k \geq 2$, Theorem 2 can be strengthened considerably, to

Theorem 3 (Mittag-Leffler Principle). *Let $k \geq 2$ and let $A(s)$ be any rational function satisfying $A(k - s) = A(s)$. Then there exists $\varphi(s)$ such that $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$ satisfies $1, 2_k$ and 3, and such that $R(s) - A(s)$ is entire.*

Finally, we can extend all of these results to the case in which 2_k is replaced by the functional equation

$$R(s) = -R(k - s). \tag{4_k}$$

Theorem 4 **(a)** *For k an arbitrary real number and $\sigma_0 \geq k/2$, let $\mathcal{B}(\sigma_0; k)$ be the space of rational functions $A(s)$ with poles restricted to the strip $k - \sigma_0 \leq \mathrm{Re}\, s \leq \sigma_0$ and satisfying $A(k - s) = -A(s)$. Let $\mathcal{B}_H(\sigma_0; k)$ denote the subspace of A in $\mathcal{B}(\sigma_0; k)$ such that $R(s) - A(s)$ is entire for some $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$ satisfying 1, 3 and 4_k. Then for $A_1, \ldots, A_n$ in $\mathcal{B}(\sigma_0; k)$ with $n > M(\sigma_0, k)$, some nontrivial linear combination of $A_1, \ldots, A_n$ lies in $\mathcal{B}_H(\sigma_0; k)$. Here $M(\sigma_0, k)$ is an explicit constant determined by σ_0 and k.*

(b) *Let $k \geq 2$ and suppose $A(s)$ is a rational function with $A(k - s) = -A(s)$. Then there exists $\varphi(s)$ such that $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$ satisfies 1, 3 and 4_k, and such that $R(s) - A(s)$ is entire.*

Theorem 1 is to be found in [12, 362, Theorem 1]. Theorem 3 appeared as Theorem 2 of the same article (362–3).

2. Brief discussion of the proofs. The first step is the observation that the abundance of Dirichlet series with a functional equation is equivalent, by the R-H-B correspondence, to the abundance of entire modular integrals on Γ_ϑ, with log-polynomial period. In this equivalence the functional equation 2_k is associated with modular integrals of weight k and multiplier system v_k^+ determined by

$$v_k^+(S^2) = 1, v_k^+(T) = e^{-\pi ik/2}, \tag{5.1}$$

while 4_k corresponds to modular integrals of weight k and multiplier system v_k^- determined by

$$v_k^-(S^2) = 1, v_k^-(T) = -e^{-\pi ik/2}. \tag{5.2}$$

The proofs, then, entail the construction of "many" modular integrals of fixed weight k on Γ_ϑ, with log-polynomial period for both multiplier systems v_k^+, v_k^-.

The construction of these modular integrals relies upon Eichler's "generalized Poincaré series" (GPS), first introduced in [4], and later developed and applied in [14,11,12,13]. To construct a GPS we must have, at the outset, a period cocycle $\{q_M\}$ on Γ_ϑ, in weight k and connected with the appropriate multiplier system, either v_k^+ or v_k^- in the present situation. We require further that $q(S^2; z) \equiv 0$ and that $q(T; z)$ is a log-polynomial sum. (Recall the alternative notation $q(M; z)$ for $q_M(z)$.) Starting with these two restrictions, we wish to generate $\{q_M\}$ by applying the cocycle condition (4.6).

As it turns out, the necessary condition

$$q_T|T + q_T = 0 \tag{5.3}$$

of §IV.3 is sufficient for this process to yield a well-defined period cocycle $\{q_M\}$. Initially, let q be any log-polynomial sum whatsoever and put $q_T = q|T - q$, which is then a log-polynomial sum satisfying (5.3). We now define a period cocycle in Γ_ϑ by writing $M \in \Gamma_\vartheta$ as a word in S^2 and T and then applying (4.6) several times to define q_M. While the relation $T^2 = I$ gives rise to the complication that M is not given uniquely as a word in S^2 and T, the restriction (5.3) on q_T guarantees the uniqueness of q_M defined in this manner.

Next, let m be a positive even integer. With $\{q_M\}$ in hand, define Eichler's *generalized poincaré series* $\Psi(\{q_M\}; m; z) = \Psi(z)$ by

$$\Psi(z) = \sum_V q_V(z)(cz+d)^{-m}; \tag{5.4}$$

the summation is on all $V = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\vartheta$ with distinct lower rows. (The condition $q(S^2; z) \equiv 0$ ensures that the individual terms of the series depend only upon the lower row c, d of $V \in \Gamma_\vartheta$.) Eichler shows that, for sufficiently large m, the series (5.4) converges absolutely for z in $\mathcal{H}$, and uniformly on compact subsets of $\mathcal{H}$ [4]. Eichler's proof assumes that the q_V are polynomials, but a slight elaboration of his method establishes convergence for the more general case $q_V \in \mathcal{P}$, $V \in \Gamma_\vartheta$ [11, 615–619]. Since we began with q_T a log-polynomial sum, it follows that $q_V \in \mathcal{P}$ for all V in Γ_ϑ, so the proof applies here.

As a consequence of absolute convergence, $\Psi(z)$ has the transformation property

$$(\Psi|M)(z) = (\gamma z+\delta)^m\Psi(z) - (\gamma z+\delta)^m E_m(z)q_M(z), \tag{5.5}$$

for $M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma_\vartheta$, where $E_m(z)$ is the familiar Eisenstein series of weight m,

$$E_m(z) = \sum_V (cz+d)^{-m} \tag{5.6}$$

with summation conditions as in (5.4). Since m is an even integer, $E_m(z)$ does not vanish identically, so we may form the quotient $F(z) = -\Psi(z)/E_m(z)$. By (5.5) and the well-known modular transformation properties of $E_m(z)$, $F(z)$ has the formal behavior of a modular integral relative to $(\Gamma_\vartheta, k, v^\pm)$, with the preassigned cocycle $\{q_M\}$ of period functions:

$$F|M = F + q_M, \quad M \in \Gamma_\vartheta. \tag{5.7}$$

On the other hand, F may not be an entire modular integral (or a modular integral at all) in the sense of §IV.2, since the zeros of $E_m(z)$ in $\mathcal{H}$ may well be poles of F. There is the further issue that F may not behave appropriately at the parabolic cusps, -1 and $i\,\infty$, of Γ_ϑ. The proofs of the Theorems are largely procedures for modifying F to obtain entire modular integrals for $\{\Gamma_\vartheta, k, v^\pm\}$.

In all of the proofs the key point is this: the log-polynomial sum q_T is completely arbitrary except for the restriction (5.3). The proof of Theorem 4 employs the multiplier system v_k^-, given in (5.2), while the proofs of Theorems 1–3 utilize v_k^+, characterized by (5.1). This is the only distinction. The proofs of those results labelled "Mittag-Leffler" rely upon a "Mittag-Leffler" theorem for modular forms of weights $k \geq 2$. This theorem fails in weights $k < 2$. For detailed proofs see [12].

VI Conclusion

1. Extension to other Hecke groups. All of the above results rely upon considerations regarding the particular Hecke triangle group $\Gamma_\vartheta = \langle S^2, T\rangle$. There is a modification of these results, in which the function $R(s) = \pi^{-s}\Gamma(s)\varphi(s)$ is replaced by

$$R_\lambda(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi_\lambda(s), \lambda > 2, \tag{6.1}$$

with $\varphi_\gamma(s)$ a Dirichlet series convergent in some right half-plane. (In the notation of (6.1), $R(s) = R_2(s)$.) Bringing to bear the Hecke triangle groups G_λ defined in (4.9) with $\lambda > 2$, leads to

Theorem 5 (Mittag-Leffler) (a) *Let k be an arbitrary real number and $A(s)$ any rational function such that $A(k-s) = A(s)$. Then there exists $\varphi_\lambda(s)$ such that $R_\lambda(s)$, defined in* (6.1), *satisfies* 1, 2_k *and* 3, *and such that $R_\lambda(s) - A(s)$ is entire.*

(b) *The same, with $A(k-s) = -A(s)$ and 2_k replaced by 4_k.*

The proof, based in part upon a Mittag-Leffler theorem for modular forms on G_λ *of all real weights*, can be found in [13].

2. Zeros. Ever since Riemann's path-breaking work on $\zeta(s)$ [16], there has been a great deal of interest in the zeros of Dirichlet series, especially those with Euler product. While we can assert nothing about Euler products for the Dirichlet series constructed here, the doctoral dissertation of Lekkerkerker [15] contains results yielding information about the distribution of their zeros in the plane. These generalize familiar results concerning the zeros of Riemann's $\zeta(s)$.

Specifically, Theorem 4 in [15, chapter II, 24] immediately implies the following for the Dirichlet series $\varphi(s)$ occurring in our Theorems 1–5: *Let $\mathcal{R}$ be a rectangle of the form*

$$|\mathrm{Re}\, s - k/2| \le \alpha, |\mathrm{Im}\, s| \le \beta,$$

with α, β real, and such that $\varphi(s)$ is holomorphic in the complement of $\mathcal{R}$. Let $N_1(T)$ $(N_2(T))$ denote the number of zeros s_0 of $\varphi(s)$ in the complement of $\mathcal{R}$ such that $0 < \mathrm{Im}\, s_0 < T(-T < \mathrm{Im}\, s_0 < 0)$. *Then,*

$$N_i(T) = \frac{1}{\pi} T \log T + \delta_\varphi T + \mathcal{O}(\log T),$$

$T \to \infty$ *for* $i = 1$ *and* 2. *Here δ_φ is a real number determined by $\varphi(s)$.*

Chapter IV of [15] presents results concerning the zeros on the "critical line" (in our notation, $\mathrm{Re}\, s = k/2$) of Dirichlet series with functional equations, but there are technical difficulties in applying them to the Dirichlet series constructed here. This matter bears further investigation.

Notes

1. (p. 1) Hecke's words are "$\varphi(s/2)$ soll in eine *irgendwo* konvergente Dirichlet-Reihe entwickelbar sein" (emphasis added). However Hamburger's proof [5] requires that $\varphi(s/2)$ be absolutely convergent for $\mathrm{Re}\, s > 1$. Siegel [17] relaxes this condition, assuming convergence for $\mathrm{Re}\, s > 2 - \theta, \theta > 0$. To my knowledge, the proof of the stronger formulation of Hamburger's version, asserted by Hecke, did not appear until 1956 [2, Theorem 7.1].
2. (p. 1) Hecke assumes that the pole at $s = 1/2$ is simple.
3. (p. 3) In hindsight, Hamburger's original proof [5] does suggest these ideas, but they are not actually present.

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